

An Existence Result for Hierarchical Stackelberg v/s Stackelberg Games

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Abstract—In Stackelberg v/s Stackelberg games a collection of leaders compete in a Nash game constrained by the equilibrium conditions of another Nash game amongst the followers. The resulting equilibrium problems are plagued by the nonuniqueness of follower equilibria and nonconvexity of leader problems whereby the problem of providing sufficient conditions for existence of global or even local equilibria remains largely open. Indeed available existence statements are restrictive and model specific. In this paper, we present what is possibly the first general existence result for equilibria for this class of games. Importantly, we impose no single-valuedness assumption on the equilibrium of the follower-level game. Specifically, under the assumption that the objectives of the leaders admit a *quasi-potential* function, a concept we introduce in this paper, the global and local minimizers of a suitably defined optimization problem are shown to be the global and local equilibria of the game. In effect existence of equilibria can be guaranteed by the solvability of an optimization problem, which holds under mild and verifiable conditions. We motivate quasi-potential games through an application in communication networks.

I. INTRODUCTION

The recent past has seen increased interest in hierarchical systems with competing participants. The standard analysis for such systems concentrates on the Stackelberg model [16] of a game – a single player, called a *leader*, acts first while all other players, called followers, act subsequently under the usual assumptions of a noncooperative game. However contemporary markets, such as those in the power industry, require the modeling and analysis of a more complex game where multiple Stackelberg leaders compete in a noncooperative game following which followers play a noncooperative game amongst themselves, taking the decisions of leaders as fixed. This situation models the clearing of a sequence of markets, such as the day-ahead and real-time markets, where the participants of the day-ahead market are taken as leaders and those of the real-time markets are taken as followers.

Let $\mathcal{N} = \{1, 2, \dots, N\}$ denote the set of leaders where leader i solves the parametrized problem:

$$\boxed{L_i(x^{-i}) \quad \min_{x_i, y_i} \varphi_i(x_i, y_i; x^{-i}) \quad \text{s.t.} \quad x_i \in X_i, y_i \in \mathcal{S}(x),}$$

$x_i \in \mathbb{R}^{m_i}$ and φ_i denote leader i 's action and objective while x^{-i} and (\bar{x}_i, x^{-i}) are defined as $x^{-i} \triangleq (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ and $(\bar{x}_i, x^{-i}) \triangleq (x_1, \dots, x_{i-1}, \bar{x}_i, x_{i+1}, \dots, x_N)$. For each x , the set of follower equilibria is denoted by $\mathcal{S}(x)$ and y_i denotes the

strategy profile of *all* followers. Though y_i is not strictly within leader i 's control, a minimization over y_i is performed by an *optimistic* leader; a *pessimistic* leader would maximize over y_i while minimizing over x_i . These notions coincide if \mathcal{S} is single-valued but can be quite different when \mathcal{S} is multi-valued. We assume that each follower solves a convex optimization problem parametrized by the strategies of the leaders and other followers. It follows that $\mathcal{S}(x)$ is given by the solution set of a variational inequality (VI), say $\text{VI}(G(x, \cdot), K(x))$; the solution set is denoted $\text{SOL}(G(x_i, x^{-i}, \cdot), K(x_i, x^{-i}))$. We assume that the set-valued map K is continuous and G is a continuous mapping of all variables.

The set X_i represents other constraints and is assumed to be a convex and compact set. For each i , objective function φ_i is defined over $X \times Y$, where $X \triangleq \prod_{i=1}^N X_i$ and Y is the ambient space of y_i for any $i \in \mathcal{N}$. We assume φ_i to be continuous for each i . Let $y = (y_1, \dots, y_N)$ and $\Omega_i(x^{-i})$ be the feasible region of $L_i(x^{-i})$, given by

$$\Omega_i(x^{-i}) \triangleq \{(x_i, y_i) \mid x_i \in X_i, y_i \in \mathcal{S}(x)\}. \quad (1)$$

Let $\Omega(x)$ denote the Cartesian product of $\Omega_i(x^{-i})$, $\Omega(x) \triangleq \prod_{i=1}^N \Omega_i(x^{-i})$ and let \mathcal{F} be defined as

$$\mathcal{F} \triangleq \{(x, y) \mid x_i \in X_i, y_i \in Y_i, y_i \in \mathcal{S}(x), i \in \mathcal{N}\}, \quad (2)$$

which is the set of tuples (x, y) such that (x_i, y_i) is feasible for $L_i(x^{-i})$ for all i . It is easily seen that \mathcal{F} is the set of fixed points of Ω , i.e. $\mathcal{F} = \{(x, y) \in \mathbb{R}^n : (x, y) \in \Omega(x, y)\}$. We denote this multi-leader multi-follower game by \mathcal{E} .

In the spirit of the Stackelberg equilibrium, or subgame perfect equilibrium, the followers' response must be taken as the Nash equilibrium of the follower-level game parametrized by leader strategy profiles, whereas the leaders themselves choose decisions their while anticipating and being constrained by the equilibrium of the follower-level game. An equilibrium of such a multi-leader multi-follower game or Stackelberg v/s Stackelberg game is the Nash equilibrium of this game between leaders.

Definition 1.1 (Global Nash equilibrium): Consider the multi-leader multi-follower game \mathcal{E} . The *global Nash equilibrium*, or simply *equilibrium*, of \mathcal{E} is a point $(x, y) \in \mathcal{F}$ that satisfies the following:

$$\varphi_i(x_i, y_i; x^{-i}) \leq \varphi_i(u_i, v_i; x^{-i}), \quad (3)$$

$\forall (u_i, v_i) \in \Omega_i(x^{-i}, y^{-i})$, and all $i \in \mathcal{N}$.

Eq (3) says that at an equilibrium (x, y) , (x_i, y_i) lies in the set of best responses to (x^{-i}, y^{-i}) for all i . The qualification “global” is useful in distinguishing the equilibrium from its stationary counterparts (referred to as a “Nash B-stationary”) or its local counterpart (referred to as a “local Nash equilibrium”). In this paper, we mainly focus on global Nash equilibria and we

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refer to them as simply “equilibria”; other notions are qualified accordingly.

Despite it being a reasonable and natural model, till date no reliable theory for the existence of equilibria to multi-leader multi-follower games is available [12]. In fact there are fairly simple multi-leader multi-follower games which admit no equilibria; a particularly telling example was shown by Pang and Fukushima [12] (we discuss this example in Section III-C). To the best of our knowledge, existence statements are known only for problems where the follower-level equilibrium is unique for each strategy profile of the leaders. These results are obtained via explicit substitution of this equilibrium and further analysis of the leader-level equilibrium [15], [17], [5]; it is evident that such a *modus operandi* can succeed only when the cost functions and constraints associated with the players take suitable simple forms. On another track, existence has been claimed for weaker notions of equilibria, e.g., solutions of the aggregated stationarity conditions of the problems of the leaders [8], [14], [11].

The analysis of such games is hindered by the inapplicability of standard fixed point theorems – in particular the lack of convexity of the leaders’ problems, and suitable continuity properties in the best response functions of the leaders. In this paper we provide a clean result for the existence of equilibria that does not assume the uniqueness or favorable structure of the follower-level equilibrium. The tool we use is similar to but distinct from the potential game [10]. We relate the minimizer of an optimization problem to the equilibrium of the game, thereby obviating the need to apply fixed point theory. Our main contributions can be summarized as follows:

(1) Global equilibria of quasi-potential games: We introduce a new class of multi-leader multi-follower games, called *quasi-potential games*, in which the leaders’ objectives take on a particular structure, part of which admits a potential function. We motivate this class of games through an application in communication networks. We show that the global minimizers of a suitably defined optimization problem are the global equilibria of such games. Consequently, sufficiency requirements for existence reduce to mild and verifiable conditions for the solvability of optimization problems (such as continuity of the objective and compactness of feasible region). Furthermore, we establish a similar relation for the pessimistic formulation.

(2) Local and Nash-Stationary Equilibria: Notably, such relationships are shown to extend to allow for relating local minimizers and stationary points of the associated optimization problem to local Nash equilibria and stationary equilibria.

The remainder of the paper is organized into three sections. In Section II, we provide an example of a quasi-potential multi-leader multi-follower game and comment on the analytical intractability of general multi-leader multi-follower games. In Section III, we provide our main results. The paper concludes in Section IV with a brief summary.

II. MULTI-LEADER MULTI-FOLLOWER GAMES: EXAMPLES AND BACKGROUND

A. Examples of multi-leader multi-follower games

In one of the first examples of multi-leader multi-follower games [15], a set of followers compete in a Cournot game to determine quantity of production, while each leader makes

a decision constrained by the equilibrium quantities produced by the followers. Furthermore, leaders compete amongst each other to decide their production levels, subject to the levels they anticipate from the followers. Following is another example from communication networks.

a) *Congestion control in communication networks:* Suppose $\mathcal{N} = \{1, \dots, N\}$ denotes the set of users and K_1, \dots, K_N are strategy sets of users. Given a set of flow decisions $x = (x_1, \dots, x_N)$ of users, the network manager solves a parametrized optimization problem given by the following:

$$\text{Net}(x) \quad \min_y f(y; x) \quad \text{s.t.} \quad y \in C(x),$$

in which y represents the network decisions (which include flow specifications etc.), $f(y; x)$ denotes the network manager’s objective and $C(x)$ represents the set of feasible allocations available to the manager for user decisions x . Each user is assumed to be a leader with respect to the network manager (follower) and the resulting user problem is given by the following:

$$L_i(x^{-i}) \max_{x_i, y} U_i(x_i) - h(y) \quad \text{s.t.} \quad x_i \in K_i, y \in \text{SOL}(\text{Net}(x)),$$

where $h(y)$ represents the congestion cost associated with the network manager’s decision y and $\text{SOL}(\dots)$ represents the solution of problem ‘...’. We assume that every user is charged the entire cost of congestion, an assumption that is standard in such models (cf. [1], [4]). Such a model represents a hierarchical generalization of the competitive model considered by Başar [4], Alpcan and Başar [1], and Yin, Shanbhag, and Mehta [18]. Notably, past work has not modeled the network manager as a separate entity but this extension has much relevance when considering the role of large, and possibly strategic, independent service providers in the context of network management. Similar models may arise in power markets where the “cost of grid reliability” is socialized [7].

B. A comment on the intractability of multi-leader multi-follower games

We now briefly comment on analytical difficulties that arise in games such as \mathcal{E} . An equilibrium of \mathcal{E} is the Nash equilibrium of the game where players solve problems $\{L_i\}_{i \in \mathcal{N}}$. At the Nash equilibrium, each player’s strategy is his “best response” assuming the strategies of his opponents are held fixed. For any tuple of strategies (x, y) , one may define a *reaction map* $\mathcal{R} : \text{dom}(\Omega) \rightarrow 2^{\text{range}(\Omega)}$ (set-valued in general) as $\mathcal{R}(x, y) := \prod_{i=1}^N \mathcal{R}_i(x^{-i}, y^{-i})$, where $\mathcal{R}_i(x^{-i}, y^{-i}) = \text{SOL}(L_i(x^{-i}))$. Although $\Omega_i(\cdot)$ is independent of y^{-i} , for this section we will write Ω_i explicitly as a function of y^{-i} with the identification $\Omega_i(x^{-i}, y^{-i}) \equiv \Omega_i(x^{-i})$. (x, y) is an equilibrium of \mathcal{E} if and only if (x, y) is a fixed point of \mathcal{R} , and one could, in principle, approach the problem through fixed point theory. However due to the nonconvexity of problems $\{L_i\}_{i \in \mathcal{N}}$ difficulties arise when one attempts to apply fixed point theorems to this reaction map. Almost all fixed point theorems rely on the following categories of assumptions:

- 1) (a) the mapping to which a fixed point is sought is assumed to be a *self-mapping*;
- 2) (b) (i) the domain of the mapping and (ii) the images are required to be of a specific shape, e.g. convex;

- 3) (c) the mapping is required to be continuous (if the mapping is single-valued) or upper semicontinuous (if set-valued).

The first difficulty encountered is that \mathcal{R} is not necessarily a self-mapping: \mathcal{R} maps $\text{dom}(\Omega)$ to $\text{range}(\Omega)$ and $\text{range}(\Omega)$ may not be a subset of $\text{dom}(\Omega)$. Second, $\text{dom}(\Omega)$ is hard to characterize and little can be said about its shape. Finally, the continuity (or upper semicontinuity) of \mathcal{R} is far from immediate. There are ways of circumventing difficulties (a) and (b), some of which have been employed in literature. If $\Omega(x, y) \neq \emptyset$ for all $(x, y) \in X \times Y^N$, where $Y^N = \prod_{i \in \mathcal{N}} Y$, we get $\text{dom}(\Omega) = X \times Y^N$ and \mathcal{R} may be taken to be a map from $X \times Y^N$ to subsets of $X \times Y^N$. This approach was employed by Arrow and Debreu [2]. If $X \times Y^N$ is convex, as is in our case, the difficulty (b(i)) is also circumvented. The (upper semi-)continuity of \mathcal{R} requires Ω to be continuous [6], a property that rarely holds if \mathcal{S} (the solution set of a VI) is multivalued. As a consequence, the shape of the mapped values ((b.ii)) and the upper semicontinuity of \mathcal{R} are the key barriers to the success of this approach. In the case where \mathcal{S} is single-valued, the continuity of \mathcal{R} follows readily (since \mathcal{S} , the solution set of a parametrized VI is upper-semicontinuous with respect to the parameter). A majority of the known results for multi-leader multi-follower games are indeed for this case.

III. QUASI-POTENTIAL MULTI-LEADER MULTI-FOLLOWER GAMES

This section contains our main results. In Section III-A, we define a new class of multi-leader multi-follower games, called *quasi-potential* games, and provide conditions for the existence of global equilibria for the optimistic and pessimistic formulation. Analogous results for local and Nash stationary equilibria, for the optimistic formulation are derived in Section III-B. Finally, in Section III-C we conclude by commenting on how the results apply to some well-known examples.

A. Existence results for global equilibria

To motivate quasi-potential games, let us first consider the case where for each tuple of leader decisions, there is a unique follower equilibrium. In this case, the leader problem L_i can be reformulated as follows:

$$\tilde{L}_i(x^{-i}) \quad \min_{x_i} \varphi_i(x_i, \mathcal{S}(x); x^{-i}) \quad \text{s.t.} \quad x_i \in X_i.$$

Denote this game as $\tilde{\mathcal{E}}$. We may now define an implicit potential multi-leader multi-follower game as follows.

Definition 3.1 (Implicit Potential multi-leader multi-follower games): An implicit potential multi-leader multi-follower game is a multi-leader multi-follower game where for each $x \in X$, $|\mathcal{S}(x)| = 1$. The i^{th} leader solves $\tilde{L}_i(x^{-i})$ and there exists a function π such that for all $i \in \mathcal{N}$, for all $(x_i, x^{-i}) \in X$, and for all $x'_i \in X_i$,

$$\begin{aligned} \varphi_i(x_i, \mathcal{S}(x); x^{-i}) - \varphi_i(x'_i, \mathcal{S}(x'_i, x^{-i}); x^{-i}) \\ = \pi(x_i, \mathcal{S}(x); x^{-i}) - \pi(x'_i, \mathcal{S}(x'_i, x^{-i}); x^{-i}). \end{aligned}$$

We thus have the following proposition; the proof is straightforward, hence we skip it.

Proposition 3.1: Consider an implicit potential multi-leader multi-follower game $\tilde{\mathcal{E}}$. Then any global minimizer of P^{imp} is

a global Nash equilibrium of $\tilde{\mathcal{E}}$, where P^{imp} is the following problem:

$$P^{\text{imp}} \quad \min_x \pi(x) \quad \text{s.t.} \quad x \in X.$$

In principle, the above is an existence result. However two challenges emerge when one considers the result in practice. First, in most cases, particularly when the second-level equilibrium constraints arises from the equilibrium conditions of a constrained problem, $\mathcal{S}(\cdot)$ is not single-valued. Second, even if $\mathcal{S}(\cdot)$ is single-valued, ascertaining the potentiality of this implicit game is difficult since it requires access to a closed-form expression for $\mathcal{S}(\cdot)$. Motivated by this, we now provide a different avenue that relies on the property of *quasi-potentiality*.

Definition 3.2 (Quasi-potential multi-leader multi-follower games):

Consider a multi-leader multi-follower game \mathcal{E} in which the player objectives are denoted by $\{\varphi_1, \dots, \varphi_N\}$. \mathcal{E} is referred to as a quasi-potential game if the following hold:

- (i) For $i \in \mathcal{N}$, there exist functions $\phi_1(x), \dots, \phi_N(x)$ and a function $h(x, y_i)$ such that each player i 's objective $\varphi_i(\cdot)$ is given as $\varphi_i(x_i, y_i; x^{-i}) \equiv \phi_i(x) + h(x, y_i)$.
- (ii) There exists a function $\pi(\cdot)$ such that for all $i = 1, \dots, N$, and for all $x \in X$ and $x'_i \in X_i$, we have $\phi_i(x_i; x^{-i}) - \phi_i(x'_i; x^{-i}) = \pi(x_i; x^{-i}) - \pi(x'_i; x^{-i})$.

The function $\pi + h$ is called the quasi-potential function.

Notice that the function h does not have a subscript ' i ', and is thereby the same for all leaders. Essentially a quasi-potential game has objective functions which can be written as a sum of an ' x ' part and an ' x, y_i ' part, wherein the ' x ' part admits a potential function in the standard sense and the ' x, y_i ' part is identical for all players. There exists a function π as required in Definition 3.2 if and only if [10] for all $x \in \prod_{i=1}^N X_i$

$$\nabla_{x_i} \pi(x) \equiv \nabla_{x_i} \phi_i(x) \quad \forall i \in \mathcal{N}. \quad (4)$$

Remark III.1. A potential multi-leader multi-follower game is one where leaders have objective functions $\varphi_i, i \in \mathcal{N}$ such that there exists a function π , called potential function, such that for all $i \in \mathcal{N}$, for all $(x_i, x^{-i}) \in X$, $(y_i, y^{-i}) \in Y$ and for all $x'_i \in X_i, y'_i \in Y_i$, $\varphi_i(x_i, y_i; x^{-i}, y^{-i}) - \varphi_i(x'_i, y'_i; x^{-i}, y^{-i}) = \pi(x_i, y_i; x^{-i}, y^{-i}) - \pi(x'_i, y'_i; x^{-i}, y^{-i})$. Notice that a quasi-potential game is a potential game, but the converse is not true. \square

Consider the optimization problem P^{quasi} , defined as:

$$P^{\text{quasi}} \quad \min_{x, w} \pi(x) + h(x, w) \quad \text{s.t.} \quad (x, w) \in \mathcal{F}^{\text{quasi}},$$

where

$$\mathcal{F}^{\text{quasi}} \triangleq \{(x, w) \mid x_i \in X_i, i \in \mathcal{N}, w \in \mathcal{S}(x)\}. \quad (5)$$

Our main result relates the minimizers of P^{quasi} to the global equilibria of \mathcal{E} .

Proposition 3.2: Consider a quasi-potential multi-leader multi-follower game \mathcal{E} . Then if (x, w) is a global minimizer of P^{quasi} , then (x, y) , where $y_i = w$ for all $i \in \mathcal{N}$, is a global equilibrium of \mathcal{E} .

Proof: Observe that a point (x_i, w) is feasible for the i^{th} agent's problem $L_i(x^{-i})$ if and only if $(x, w) \in \mathcal{F}^{\text{quasi}}$. i.e., for each i

$$(x_i, w) \in \Omega_i(x^{-i}) \iff (x, w) \in \mathcal{F}^{\text{quasi}}. \quad (6)$$

Now suppose, (x, w) is a solution of P^{quasi} . Then,

$$\pi(x) + h(x, w) \leq \pi(x') + h(x', w'), \forall x' \in X$$

and $\forall w' \in S(x')$. More specifically, taking $x' = (x'_i, x^{-i})$ and $w' \in S(x'_i, x^{-i})$, and using (6), it follows that

$$\pi(x_i; x_{-i}) + h(x, w) \leq \pi(x'_i; x_{-i}) + h((x'_i, x_{-i}), w'),$$

$\forall (x'_i, w') \in \Omega_i(x^{-i})$. Since \mathcal{E} is a quasi-potential game,

$$\pi(x_i, x_{-i}) - \pi(x'_i, x_{-i}) = \phi_i(x_i; x_{-i}) - \phi_i(x'_i, x_{-i}).$$

Let y be such that $y_i = w$ for all $i \in \mathcal{N}$. As a result $\varphi_i(x_i, y_i; x^{-i}) \leq \varphi_i(x'_i, w'; x^{-i})$, $\forall x'_i, w' \in \Omega_i(x^{-i})$, and hence (x_i, y_i) is a solution $L_i(x^{-i})$. The result follows. ■

Given this relationship between the minimizers of P^{quasi} and the equilibria of \mathcal{E} , existence of equilibria is guaranteed by the solvability of P^{quasi} , as formalized by the next result.

Theorem 3.3 (Existence of global equilibria of \mathcal{E}): Let \mathcal{E} be a quasi-potential multi-leader multi-follower game. Suppose $\mathcal{F}^{\text{quasi}}$ is a nonempty set and φ_i is a continuous function for $i = 1, \dots, N$. If the minimizer of P^{quasi} exists (for example, if either π is a coercive function over $\mathcal{F}^{\text{quasi}}$ or if $\mathcal{F}^{\text{quasi}}$ is compact), then \mathcal{E} admits an equilibrium.

Observe that $\mathcal{F}^{\text{quasi}}$ is a closed set if K, G are continuous: the constraint $w \in S(x)$ is equivalent to a nonlinear equation $\mathbf{F}^{\text{nat}}(w; x) = 0$, where $\mathbf{F}^{\text{nat}}(\cdot; x)$ is the natural map [6] of $\text{VI}(G(\cdot; x), K(x))$, given by $\mathbf{F}^{\text{nat}}(y; x) = y - \Pi_{K(x)}(y - G(y; x))$, where $\Pi_Z(z)$ is the projection of z on a set Z . By the continuity of \mathbf{F}^{nat} the zeros of \mathbf{F}^{nat} form a closed set. Consequently, if K, G are continuous (which would hold, e.g., when the follower objectives are continuously differentiable and their constraints are independent of x), then ascertaining the compactness of $\mathcal{F}^{\text{quasi}}$ amounts to only its boundedness. More generally, P^{quasi} is a mathematical program with equilibrium constraints and under coercivity of the objective or compactness of the feasible region, a global minimizer exists [9, Ch. 1].

Consider a special case of \mathcal{E} , denoted \mathcal{E}^{ind} , where the i^{th} leader solves the following problem wherein the objective is independent of y_i .

$$L_i^{\text{ind}}(x^{-i}) \min_{x_i, y_i} \varphi_i(x_i; x^{-i}) \text{ s.t. } x_i \in X_i, y_i \in S(x).$$

The following corollary captures the relationship between the global minimizers of P^{quasi} and the global equilibria of \mathcal{E}^{ind} .

Corollary 3.4: Consider game \mathcal{E}^{ind} in which for each i , $\varphi_i(x_i, y_i; x^{-i}) \equiv \varphi_i(x_i, x^{-i})$ and the functions $\{\varphi_i\}_{i \in \mathcal{N}}$ admit a potential function. Then this game is a quasi-potential multi-leader multi-follower game. Further, if (x, w) is a global minimizer of P^{quasi} , then (x, y) where $y_i = w$ for all $i \in \mathcal{N}$ is a global equilibrium of \mathcal{E}^{ind} . If a solution exists to P^{quasi} , it is a global equilibrium of \mathcal{E}^{ind} .

The proof of this corollary follows from noting that \mathcal{E}^{ind} is trivially a quasi-potential game (in Definition 3.2, take $h \equiv 0$ and π to be the given potential function of $\{\varphi_i\}_{i \in \mathcal{N}}$).

At this juncture, it is worth differentiating the above existence statements from more standard results presented in [15], [17] where the follower equilibrium decisions are eliminated by leveraging the single-valuedness of the solution set of the follower equilibrium problem. In these approaches, the final claim rests on showing that the implicitly defined objective function (in the

x -space) is convex and continuous, properties that again require further assumptions. In comparison, we do not impose any such requirement. Finally, we believe that the class of *quasi-potential games* is not an artificial construct. For instance, the congestion control games arising in communication networks in II-A lead to a quasi-potential multi-leader multi-follower game.

Remark III.2. One may ask the following question: if the objectives of the leaders admit a potential function in the x, y space, as in Remark III.1, then does the resulting game have an equilibrium? The answer is no, as demonstrated by an example of Pang and Fukushima [12]. We consider this example in Section III-C. This poses a challenge for generalizing our results beyond the class of quasi-potential games. □

1) Pessimistic formulation: In this formulation, the i^{th} leader solves the following problem:

$$\underline{L}_i(x^{-i}) \min_{x_i} \max_{y_i} \varphi_i(x_i, y_i; x^{-i}) \text{ s.t. } x_i \in X_i, y_i \in S(x)$$

Denote the resulting game by $\underline{\mathcal{E}}$; an equilibrium of $\underline{\mathcal{E}}$ is defined as a Nash equilibrium of $\{\underline{L}_i\}_{i \in \mathcal{N}}$. We know of no existence results for equilibria of the pessimistic formulation. Indeed, solutions to individual leader problems \underline{L}_i even may not exist since each \underline{L}_i is effectively a *trilevel* optimization problem (the problem that defines $S(\cdot)$, which is nested inside the maximization over y_i , which in turn is nested inside the minimization over x_i); for such problems, strong continuity properties of inner nested problems are as good as necessary [6] for the problem to admit a solution. Of course, when $S(\cdot)$ is single-valued, the pessimistic and optimistic formulations coincide and thereby the Theorem 3.3 applies to $\underline{\mathcal{E}}$. Nevertheless, if the leader objectives $\varphi_i, i \in \mathcal{N}$ admit a quasi-potential function, one can obtain a relation between the solution, if it exists, of a problem analogous to P^{quasi} and the equilibrium of $\underline{\mathcal{E}}$. Consider the problem

$$\underline{P}^{\text{quasi}} \min_{x_i} \max_w \pi(x) + h(x, w) \text{ s.t. } x \in X, w \in S(x).$$

Theorem 3.5: Consider the game $\underline{\mathcal{E}}$ and suppose that the objectives of the leaders admit a quasi-potential function. If (x, w) is a solution of $\underline{P}^{\text{quasi}}$ then (x, y) where $y_i = w$ for all $i \in \mathcal{N}$ is an equilibrium of $\underline{\mathcal{E}}$. Consequently, if $\underline{P}^{\text{quasi}}$ admits a solution, $\underline{\mathcal{E}}$ admits a solution.

Proof: Let (x, w) solve $\underline{P}^{\text{quasi}}$. Therefore, clearly,

$$\pi(x) + \max_{w \in S(x)} h(x, w) \leq \pi(x') + \max_{w' \in S(x')} h(x', w') \quad \forall x' \in X.$$

Let $i \in \mathcal{N}$ and $\bar{x}_i \in X_i$ be arbitrary and put $x' = (\bar{x}_i, x^{-i})$ in the inequality above to get

$$\pi(x) + \max_{w \in S(x)} h(x, w) \leq \pi(\bar{x}_i, x^{-i}) + \max_{w' \in S(\bar{x}_i, x^{-i})} h(\bar{x}_i, x^{-i}, w')$$

$\forall \bar{x}_i \in X_i$. Since $\underline{\mathcal{E}}$ is a quasi-potential game, we get

$$\begin{aligned} \phi_i(x) + \max_{w \in S(x)} h(x, w) &\leq \phi_i(\bar{x}_i, x^{-i}) \\ &+ \max_{w' \in S(\bar{x}_i, x^{-i})} h(\bar{x}_i, x^{-i}, w') \quad \forall \bar{x}_i \in X_i. \end{aligned}$$

Since this holds for each i , it follows that (x, y) where $y_i = w \quad \forall i \in \mathcal{N}$ is an equilibrium of $\underline{\mathcal{E}}$. ■

Remark III.3. $\underline{P}^{\text{quasi}}$ represents an instance of a Stackelberg equilibrium problem and its solvability has been studied in several places, e.g., [3]. □

B. Local and Nash stationary equilibria

While the discussion thus far provides an approach for claiming existence of global equilibria by obtaining a global solution to a suitable optimization problem. However, the computation of a global minimizer of P^{quasi} is a difficult nonconvex problem that falls within the category of mathematical programs with equilibrium constraints (MPEC). However, one can often obtain stationary points or local minimizers of such problems and in this subsection, we relate these points to analogous local or stationarity variants of Nash equilibria. We begin with a formal definition of a Nash Bouligand stationary or a Nash B-stationary point.¹

Definition 3.3 (Nash B-stationary point): A point $(x, y) \in \mathcal{F}$ is a Nash B-stationary point of \mathcal{E} if for all $i \in \mathcal{N}$, $\nabla_i \varphi_i(x, y)^\top d \geq 0$, $\forall d \in \mathcal{T}((x_i, y_i); \Omega_i(x^{-i}))$, where $\mathcal{T}(z; K)$, the tangent cone at $z \in K \subseteq \mathbb{R}^n$, is defined as follows:

$$\mathcal{T}(z; K) \triangleq \{dz \in \mathbb{R}^n : \exists 0 < \tau_k \rightarrow 0, K \ni \{z_k\} \rightarrow z \text{ such that } dz = \lim_{k \rightarrow \infty} ((z_k - z)/\tau_k)\}.$$

Proposition 3.6 (Nash B-stationary points of \mathcal{E}): Consider a quasi-potential multi-leader multi-follower game \mathcal{E} and suppose φ_i is a continuously differentiable function over $X \times Y$ for $i = 1, \dots, N$. If (x, w) is a B-stationary point of P^{quasi} , then (x, y) where $y_i = w$ for all $i \in \mathcal{N}$ is a Nash B-stationary point of \mathcal{E} .

Proof: A stationary point (x, w) of P^{quasi} satisfies

$$\nabla_x(\pi(x) + h(x, w))^\top dx + \nabla_w h(x, w)^\top dw \geq 0, \quad (7)$$

$\forall (dx, dw) \in \mathcal{T}((x, w); \mathcal{F}^{\text{quasi}})$. Fix an $i \in \mathcal{N}$ and consider an arbitrary $(dx'_i, dy'_i) \in \mathcal{T}(x_i, w; \Omega_i(x^{-i}))$. By the definition of the tangent cone, there exists a sequence $\Omega(x^{-i}) \ni (u_{i,k}, v_{i,k}) \xrightarrow{k} (x_i, w)$ and a sequence $0 < \tau_k \xrightarrow{k} 0$ such that $\frac{u_{i,k} - x_i}{\tau_k} \xrightarrow{k} dx'_i$ and $\frac{v_{i,k} - w}{\tau_k} \xrightarrow{k} dy'_i$. It follows that the sequence $(\mathbf{x}_{i,k}, \mathbf{y}_{i,k})$, where $\mathbf{x}_{i,k} = (x_1, \dots, u_{i,k}, \dots, x_N)$, and $\mathbf{y}_{i,k} = v_{i,k}$, satisfies $(\mathbf{x}_{i,k}, \mathbf{y}_{i,k}) \in \mathcal{F}^{\text{quasi}}$. Therefore, the direction $(\mathbf{dx}_i, \mathbf{dw})$ where $\mathbf{dx}_i = (0, \dots, dx'_i, \dots, 0)$ and $\mathbf{dy}_i = dy'_i$, belongs to $\mathcal{T}((x, w); \mathcal{F}^{\text{quasi}})$. Substituting $(dx, dw) = (\mathbf{dx}_i, \mathbf{dy}_i)$ in (7) and using (4) gives

$$\nabla_{x_i}(\phi_i(x) + h(x, w))^\top dx'_i + \nabla_w h(x, w)^\top dy'_i \geq 0.$$

Since, $i \in \mathcal{N}$ and $(dx'_i, dy'_i) \in \mathcal{T}(x_i, w; \Omega_i(x^{-i}))$ were arbitrary, (x, w) is a Nash B-stationary point of \mathcal{E} . ■

We now define a local Nash equilibrium and show its relationship to the local minimum of P^{quasi} .

Definition 3.4 (Local Nash equilibrium): A point $(x, y) \in \mathcal{F}$ is a local Nash equilibrium of \mathcal{E} if for all $i \in \mathcal{N}$, (x_i, y_i) is a local minimum of $L_i(x^{-i})$.

Proposition 3.7 (Local Nash equilibrium of \mathcal{E}): Consider a quasi-potential multi-leader multi-follower game \mathcal{E} and suppose φ_i is continuously differentiable function over $X \times Y$ for $i = 1, \dots, N$. If (x, w) is a local minimizer of P^{quasi} , then (x, y) , where $y = (w, \dots, w)$ is a local Nash equilibrium of \mathcal{E} .

Proof: If (x, w) is a local minimum of P^{quasi} , there exists a neighborhood of (x, w) , denoted by $\mathcal{B}(x, w)$, such that

$$\pi(x) + h(x, w) \leq \pi(x') + h(x', w'), \quad (8)$$

¹A primal-dual characterization of B-stationarity is provided by Pang and Fukushima [13].

for all $(x', w') \in \mathcal{B}(x, w) \cap \mathcal{F}^{\text{quasi}}$. Consider an arbitrary $i \in \mathcal{N}$ and let $\mathcal{B}_i(x_i, w; x^{-i}) := \{(u_i, v_i) \mid (u_i, x^{-i}, v_i) \in \mathcal{B}(x, w)\}$. Then by the definition of $\mathcal{F}^{\text{quasi}}$ in (5) it follows that

$$(u_i, v_i) \in \Omega_i(x^{-i}) \cap \mathcal{B}_i(x_i, w; x^{-i}) \\ \iff (u_i, x^{-i}, v_i) \in \mathcal{F}^{\text{quasi}} \cap \mathcal{B}(x, w).$$

Thus, in (8) put $x' = (u_i, x^{-i})$, $w' = v_i$ to get

$$\pi(x) + h(x, w) \leq \pi(u_i, x^{-i}) + h(u_i, x^{-i}, v_i),$$

$\forall (u_i, v_i) \in \Omega_i(x^{-i}) \cap \mathcal{B}_i(x_i, w; x^{-i})$. Then employing (III.1), we get

$$\varphi_i(x, w) \leq \varphi_i(u_i, x^{-i}, v_i), \quad \forall (u_i, v_i) \in \Omega_i(x^{-i}) \cap \mathcal{B}_i(x_i, w; x^{-i}).$$

In other words, (x_i, w) is a local minimizer of $L_i(x^{-i})$. This holds for each $i \in \mathcal{N}$, whereby (x, y) where $y = (w, \dots, w)$ is a local Nash equilibrium. ■

For sake of brevity we have chosen to focus only on the Nash B-stationary points and local Nash equilibria. Similar relationships hold also between other notions of stationarity.

C. Revisiting the example of Pang and Fukushima [12]

In 2005, Pang and Fukushima [12] presented an example of a simple multi-leader multi-follower game that had no equilibrium. This game has two leaders with objectives φ_1 and φ_2 , defined as follows: $\varphi_1(x_1, y_1) = \frac{1}{2}x_1 + y_1$ and $\varphi_2(x_2, y_2) = -\frac{1}{2}x_2 - y_2$. This game admits a potential function in the x, y space, but the game is not a quasi-potential game. However, several variants of this game with a modified φ_2 have an equilibrium, since they turn out to be quasi-potential games. For example, in a particular variant, the leader objectives are given as follows: $\varphi_1(x_1, y_1) = \frac{1}{2}x_1 + y_1$ and $\varphi_2(x_2, y_2) = -\frac{1}{2}x_2 + y_2$, and this game has an equilibrium. We discuss the original example and the variants next.

Example III.1. Pang and Fukushima [12] In this example, $\mathcal{N} = \{1, 2\}$, $X_1 = X_2 = [0, 1]$ and $Y = \mathbb{R}$, and there is one follower [12] which solves the optimization problem

$$\min_{y \geq 0} \{y(-1 + x_1 + x_2) + \frac{1}{2}y^2\} = \max \{0, 1 - x_1 - x_2\}$$

Thus the problems L_1, L_2 are as follows.

$L_1(x_2) \min_{x_1, y_1} \frac{1}{2}x_1 + y_1 \quad \text{s.t.} \quad \begin{array}{l} x_1 \in [0, 1], \\ y_1 = \max \{0, 1 - x_1 - x_2\}. \end{array}$
$L_2(x_1) \min_{x_2, y_2} -\frac{1}{2}x_2 - y_2 \quad \text{s.t.} \quad \begin{array}{l} x_2 \in [0, 1], \\ y_2 = \max \{0, 1 - x_1 - x_2\}. \end{array}$

It can be easily checked that this game has no equilibrium. We refer the reader to [12] for details.

Observe that this game admits a potential function in the (x, y) space given readily by $\pi(x, y) = \varphi_1(x_1, y_1) + \varphi_2(x_2, y_2) = \frac{1}{2}x_1 + y_1 - \frac{1}{2}x_2 - y_2$. However this game is *not* a quasi-potential game, since, clearly, one cannot find a function h such that $h(x, y_1) = y_1$ and $h(x, y_2) = -y_2$ to meet the requirement of Definition 3.2. We now consider a modification of this example.

a) *Quasi-potential variants of Pang and Fukushima [12]:*

Consider the following variant of the Pang and Fukushima example: take $\varphi_1(x_1, y_1) = \frac{1}{2}x_1 + h(y_1)$, $\varphi_2(x_2, y_2) = -\frac{1}{2}x_2 + h(y_2)$, where $h(\cdot)$ is a continuous function. This game is a quasi-potential game, with quasi-potential function given by $\pi(x) + h(y) = \frac{1}{2}x_1 - \frac{1}{2}x_2 + h(y)$. By Proposition 3.2, a global minimizer of P^{quasi} is a global equilibrium of the \mathcal{E} , where P^{quasi} is defined as

$P^{\text{quasi}} \quad \min_{x, w} \quad \frac{1}{2}x_1 - \frac{1}{2}x_2 + h(w)$
$\text{s.t.} \quad w = \max \{0, 1 - x_1 - x_2\},$ $x_1, x_2 \in [0, 1].$

Let us consider some special cases of h .

Take $h(w) \equiv w$. In this case, in L_1, L_2 , one may substitute y_1 and y_2 , resulting in leader problems \tilde{L}_1, \tilde{L}_2 in x_1, x_2 , with objectives, $\tilde{\varphi}_1(x_1, x_2) = \frac{1}{2}x_1 + \max\{0, 1 - x_1 - x_2\}$, $\tilde{\varphi}_2(x_1, x_2) = -\frac{1}{2}x_1 + \max\{0, 1 - x_1 - x_2\}$, respectively. Notice that $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are convex in x_1 and x_2 respectively. Since the feasible regions of both problems are convex and compact (they are unit intervals) and the objective is convex and continuous, classical results suffice for claiming [3] that this game has an equilibrium.

Take $h(w) \equiv -w$. In this case, again one may substitute for y_1, y_2 in terms of x_1, x_2 . The resulting problems \tilde{L}_1 and \tilde{L}_2 are nonconvex. Notice that P^{quasi} is equivalent to minimizing $\frac{1}{2}x_1 - \frac{1}{2}x_2 - \max\{0, 1 - x_1 - x_2\}$ over $\{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2\}$. It can be observed that the minimizer of P^{quasi} is given by $(x_1, x_2, w) = (0, 0, 1)$. To see why the point $(x_1, x_2, y_1, y_2) = (0, 0, 1, 1)$ is an equilibrium, notice that given $x_2 = 0, y_2 = 1$, the global minimizer of $L_1(x_2, y_2)$ is $x_1 = 0, y_1 = 1$. Similarly, with $x_1 = 0, y_1 = 1$, the global minimizer of $L_2(x_1, y_1)$ is again given by $x_2 = 0, y_2 = 1$. \square

Thus we see that quasi-potentiality is a powerful property that allows one to leverage the unique structure of multi-leader multi-follower games to claim existence of equilibria.

IV. CONCLUSIONS

We consider multi-leader multi-follower games and examine the question of the existence of an equilibrium. A standard approach requires ascertaining when the reaction map admits fixed points. However, this avenue has several hindrances, an important one being the lack of continuity in the solution set associated with the equilibrium constraints capturing the follower equilibrium. We observed that these challenges can be circumvented for *quasi-potential* multi-leader multi-follower games. We show that any global minimizer of a suitably defined optimization problem is a global equilibrium of the game and that a similar result holds for the pessimistic formulation. Consequently, the above results reduced a question of the existence of an equilibrium to that of the solvability of an optimization problem, which can be claimed under fairly standard conditions that are tractable and verifiable – e.g., coercive objective over a nonempty feasible region – and the existence of a global equilibrium was seen to follow. We further showed that local minima and B-stationary points of the respective MPECs are local Nash equilibria and Nash B-stationary points of the corresponding multi-leader multi-follower game.

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